

EXACT UNIFICATION

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ABSTRACT. A new hierarchy of “exact” unification types is introduced, motivated by the study of admissibility for equational classes and non-classical logics. In this setting, unifiers of identities in an equational class are pre-ordered, not by instantiation, but rather by inclusion over the corresponding sets of unified identities. Minimal complete sets of unifiers under this new preordering always have a smaller or equal cardinality than those provided by the standard instantiation preordering, and in significant cases a dramatic reduction may be observed. In particular, the classes of distributive lattices, idempotent semigroups, and MV-algebras, which all have nullary unification type, have unitary or finitary exact type. These results are obtained via an algebraic interpretation of exact unification, inspired by Ghilardi’s algebraic approach to equational unification.

1. INTRODUCTION

It has long been recognized that the study of admissible rules is inextricably bound up with the study of equational unification (see, e.g., [28, 14, 15]). Indeed, from an algebraic perspective, admissibility in an equational class (variety) of algebras may be viewed as a generalization of unifiability in that class, and conversely, checking admissibility may be reduced to comparing certain sets of unifiers. This paper provide a new classification of equational unification problems that simplifies such reductions.¹

Let us fix an equational class of algebras \mathcal{V} for a language \mathcal{L} and denote by $\mathbf{Fm}_{\mathcal{L}}(X)$, the *formula algebra* (absolutely free algebra or term algebra) of \mathcal{L} over a set of variables $X \subseteq \omega$. A substitution (homomorphism) $\sigma: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ is called a \mathcal{V} -*unifier* (over X) of a set of \mathcal{L} -identities Σ with variables in X if

$$\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi) \quad \text{for all } \varphi \approx \psi \text{ in } \Sigma.$$

A clause $\Sigma \Rightarrow \Delta$ (an ordered pair of finite sets of \mathcal{L} -identities Σ, Δ) is \mathcal{V} -*admissible* if for each substitution $\sigma: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ where the variables in $\Sigma \cup \Delta$ are contained in X ,

$$\sigma \text{ is a } \mathcal{V}\text{-unifier of } \Sigma \quad \Rightarrow \quad \sigma \text{ is a } \mathcal{V}\text{-unifier of some member of } \Delta.$$

In particular, Σ is \mathcal{V} -unifiable if and only if $\Sigma \Rightarrow \emptyset$ is not \mathcal{V} -admissible.

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¹The reader is referred to [6] and [23] for basic notions of universal algebra and category theory, respectively.

Now suppose that the unification type of \mathcal{V} is at most finitary, meaning that every \mathcal{V} -unifier of a set of \mathcal{L} -identities Σ over the variables in Σ is a substitution instance of one of a finite set S of \mathcal{L} -unifiers of Σ . Then a clause $\Sigma \Rightarrow \Delta$ is \mathcal{V} -admissible if each member of S is an \mathcal{L} -unifier of a member of Δ . If there is an algorithm for determining the finite basis set S for Σ and the equational theory of \mathcal{V} is decidable, then checking \mathcal{V} -admissibility is also decidable. This observation, together with the pioneering work of Ghilardi on equational unification for classes of Heyting and modal algebras [14, 15], has led to a wealth of decidability, complexity, and axiomatization results for admissibility in these classes and corresponding modal and intermediate logics [16, 17, 19, 11, 4, 3, 26, 22].

The success of this approach to admissibility appears to rely on considering varieties with at most finitary unification type. This is not a necessary condition, however, as illustrated by the case of MV-algebras, the algebraic semantics of Łukasiewicz infinite-valued logic. Decidability, complexity, and axiomatization results for admissibility in this class have been established by Jeřábek [20, 21, 22] via a similar reduction of finite sets of identities to finite approximating sets of identities. On the other hand, it has been shown by Marra and Spada [25] that the class of MV-algebras has nullary unification type. This means in particular that there are finite sets of identities for which no finite basis of unifiers exists. Further examples of this discrepancy may be found in [10], including the very simple example of the class of distributive lattices where admissibility and validity of clauses coincide but unification is nullary.

As mentioned above, it is possible to check the \mathcal{V} -admissibility of a clause $\Sigma \Rightarrow \Delta$ by checking that every \mathcal{V} -unifier of Σ in a certain “basis set” \mathcal{V} -unifies Δ . Such a basis set S typically has the property that every other \mathcal{V} -unifier of Σ is a substitution instance of a member of S . The starting point for this paper is the observation that a weaker condition on S suffices, leading potentially to smaller sets of \mathcal{V} -unifiers. What is really required for checking admissibility is the property that each \mathcal{V} -unifier of Σ is also a \mathcal{V} -unifier of all identities \mathcal{V} -unified by some particular member of S . Then $\Sigma \Rightarrow \Delta$ is \mathcal{V} -admissible if each member of S is a \mathcal{V} -unifier of a member of Δ . This leads to a new ordering of \mathcal{V} -unifiers and hierarchy of exact (unification) types.

We obtain also a Ghilardi-style algebraic characterization of exact unification, where the role of formulas is taken by the finitely presented algebras of the equational class. In Ghilardi’s approach, a unifier is a homomorphism from a finitely presented algebra into a projective algebra of the class, and unifiers are preordered by composition of homomorphisms. Here, coexact unifiers are defined as homomorphisms from a finitely presented algebra onto an exact algebra (an algebra that embeds into the free algebra of the class on countably infinitely many generators) and the preordering remains the same. This contrasts with the syntactic approach to exact unification where the unifiers remain unchanged but a new preorder is introduced. Nevertheless, the syntactic and algebraic exact unification types coincide as in the standard approach.

Although certain equational classes have the same exact type as unification type (in particular, any equational class of unitary type), crucially we obtain examples where the exact type is smaller. In particular, distributive lattices have unitary exact type, while idempotent semigroups, various classes of pseudo-complemented distributive lattices, and MV-algebras have finitary exact type. We also provide

an example (due to R. Willard) of an equational class of infinitary unification type but finitary exact type.

We proceed as follows. In Section 2, we recall standard notions of equational unification and admissible rules, and Ghilardi's algebraic account of unification types. In Section 3, we introduce the new notion of exact unifier and exact unification types, providing an algebraic interpretation and applications in Section 4. Several cases studies are considered in Section 5 and some ideas for further research are presented in Section 6.

2. EQUATIONAL UNIFICATION AND ADMISSIBILITY

In this section, we describe briefly some key ideas from the theory of equational unification (referring to [2] for further details) and their relevance to the study of admissible rules. We recall, in particular, the unification type of a finite set of identities in an equational class and the algebraic interpretation of unification types provided by Ghilardi in [13]. These ideas, and also developments in subsequent sections, are most elegantly presented in the general setting of preordered sets.

Let $\mathbf{P} = \langle P, \leq \rangle$ be a preordered set (i.e., \leq is a reflexive and transitive binary relation on P). A *complete* set for \mathbf{P} is a subset $M \subseteq P$ such that for every $x \in P$, there exists $y \in M$ satisfying $x \leq y$. A complete set M for \mathbf{P} is called a μ -set for \mathbf{P} if $x \not\leq y$ and $y \not\leq x$ for all distinct $x, y \in M$. It is easily seen that if \mathbf{P} has a μ -set, then every μ -set of \mathbf{P} has the same cardinality. Hence \mathbf{P} may be said to be *nullary* if it has no μ -sets ($\text{type}(\mathbf{P}) = 0$), *infinitary* if it has a μ -set of infinite cardinality ($\text{type}(\mathbf{P}) = \infty$), *finitary* if it has a finite μ -set of cardinality greater than 1 ($\text{type}(\mathbf{P}) = \omega$), and *unitary* if it has a μ -set of cardinality 1 ($\text{type}(\mathbf{P}) = 1$). These types are ordered as follows: $1 < \omega < \infty < 0$.

The following useful lemma demonstrates that the type of a preordered set may be viewed as a categorical invariant.

Lemma 1. *Suppose that two preordered sets $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ are equivalent: i.e., there exists a map $e: P \rightarrow Q$ such that*

- (1) *for each $q \in Q$, there is a $p \in P$ such that $e(p) \leq q$ and $q \leq e(p)$*
- (2) *for each $p_1, p_2 \in P$, $p_1 \leq p_2$ iff $e(p_1) \leq e(p_2)$.*

Then $\langle P, \leq \rangle$ and $\langle Q, \leq \rangle$ have the same type.

We turn our attention now to the syntactic account of equational unification. Let us fix \mathcal{L} to be an algebraic language and \mathcal{V} an equational class of \mathcal{L} -algebras (equivalently, a variety: a class of \mathcal{L} -algebras closed under taking products, subalgebras, and homomorphic images).² Let $X \subseteq \omega$ be a set of variables, and consider substitutions $\sigma_i: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ for $i = 1, 2$. We say that σ_1 is *more general* than σ_2 (written $\sigma_2 \preceq \sigma_1$) if there exists a substitution $\sigma': \mathbf{Fm}_{\mathcal{L}}(\omega) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ such that $\sigma' \circ \sigma_1 = \sigma_2$.

Let Σ be a finite set of \mathcal{L} -identities, denoting the variables occurring in Σ by $\text{Var}(\Sigma)$. Then $\text{U}_{\mathcal{V}}(\Sigma)$ is defined as the set of \mathcal{V} -unifiers of Σ over $\text{Var}(\Sigma)$ preordered by \preceq . For $\text{U}_{\mathcal{V}}(\Sigma) \neq \emptyset$, the \mathcal{V} -unification type of Σ is defined as $\text{type}(\text{U}_{\mathcal{V}}(\Sigma))$. The *unification type of \mathcal{V}* is the maximal type of a \mathcal{V} -unifiable finite set Σ of \mathcal{L} -identities.

²The results of this paper are also valid for quasi-equational classes and, more generally, for pre-varieties (classes of algebras closed under products, subalgebras and isomorphic images). However, as all of our examples and the vast majority of cases considered in the literature are equational classes, we restrict our account to this slightly simpler setting.

Example 2. *Equational unification has been studied for a wide range of equational classes. In the most general setting of syntactic unification where \mathcal{V} is the class of all \mathcal{L} -algebras, every syntactically unifiable finite set Σ of \mathcal{L} -identities has a most general unifier; that is, syntactic unification is unitary (see, e.g., [2]). The class of Boolean algebras is also unitary [7]: if $\{\varphi \approx \top\}$ is unifiable (equivalent to the satisfiability of φ), then it has a most general unifier defined by $\sigma(x) = \neg\varphi \vee x$ for each $x \in \text{Var}(\varphi)$. The class of Heyting algebras is not unitary; for example, $\{x \vee y \approx \top\}$ has a μ -set of unifiers $\{\sigma_1, \sigma_2\}$ where $\sigma_1(x) = \top$, $\sigma_1(y) = y$, $\sigma_2(x) = x$, $\sigma_2(y) = \top$. It is, however, finitary [14]. More problematically, the class of semigroups is infinitary [27]: e.g., $\{x \cdot y \approx y \cdot x\}$ has a μ -set $\{\sigma_{m,n} \mid \gcd(m,n) = 1\}$ where $\sigma_{m,n}(x) = z^m$ and $\sigma_{m,n}(y) = z^n$. Many familiar classes of algebras are nullary; in particular, the class of distributive lattices has nullary unification type (see [14]); e.g., $\{x \wedge y \approx z \vee w\}$ has no μ -set. Other nullary classes of algebras include idempotent semigroups (bands) [1], pseudo-complemented distributive lattices [13], and MV-algebras [25].*

We now recall Ghilardi's algebraic account of equational unification [13]. Let $\mathbf{F}_{\mathcal{V}}(X)$ denote the free \mathcal{L} -algebra of \mathcal{V} over a set of variables X and let $h_{\mathcal{V}}: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{F}_{\mathcal{V}}(X)$ be the canonical homomorphism (that is, the unique homomorphism that acts as the identity on the elements of X). Given a finite set of \mathcal{L} -identities Σ and a finite set $X \supseteq \text{Var}(\Sigma)$, we denote by $\mathbf{Fp}_{\mathcal{V}}(\Sigma, X)$ the algebra in \mathcal{V} finitely presented by Σ and X : that is, the quotient algebra $\mathbf{F}_{\mathcal{V}}(X)/\Theta_{\Sigma}$ where Θ_{Σ} is the congruence on $\mathbf{F}_{\mathcal{V}}(X)$ generated by the set $\{(h_{\mathcal{V}}(\varphi), h_{\mathcal{V}}(\psi)) \mid \varphi \approx \psi \in \Sigma\}$. We also let $\text{FP}(\mathcal{V})$ denote the class of finitely presented algebras of \mathcal{V} .

Given $\mathbf{A} \in \text{FP}(\mathcal{V})$, a homomorphism $u: \mathbf{A} \rightarrow \mathbf{B}$ is called a *unifier* for \mathbf{A} if $\mathbf{B} \in \text{FP}(\mathcal{V})$ is *projective* in \mathcal{V} : that is, there exist homomorphisms $\iota: \mathbf{B} \rightarrow \mathbf{F}_{\mathcal{V}}(\omega)$ and $\rho: \mathbf{F}_{\mathcal{V}}(\omega) \rightarrow \mathbf{B}$ such that $\rho \circ \iota$ is the identity map on \mathbf{B} . Let $u_i: \mathbf{A} \rightarrow \mathbf{B}_i$ for $i = 1, 2$ be unifiers for \mathbf{A} . Then u_1 is *more general than* u_2 , written $u_2 \leq u_1$, if there exists a homomorphism $f: \mathbf{B}_1 \rightarrow \mathbf{B}_2$ such that $f \circ u_1 = u_2$.

Let $\text{U}_{\mathcal{V}}(\mathbf{A})$ be the set of unifiers of $\mathbf{A} \in \text{FP}(\mathcal{V})$ preordered by \leq . For $\text{U}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$, the *unification type of \mathbf{A} in \mathcal{V}* is defined as $\text{type}(\text{U}_{\mathcal{V}}(\mathbf{A}))$ and the *algebraic unification type of \mathcal{V}* is the maximal type of \mathbf{A} in $\text{FP}(\mathcal{V})$ such that $\text{U}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$.

Theorem 3 (Ghilardi [13]). *Let Σ be a \mathcal{V} -unifiable finite set of identities and let \mathbf{A} denote the finitely presented algebra $\mathbf{Fp}_{\mathcal{V}}(\Sigma, \text{Var}(\Sigma))$. Then*

$$\text{type}(\text{U}_{\mathcal{V}}(\Sigma)) = \text{type}(\text{U}_{\mathcal{V}}(\mathbf{A})).$$

Hence the algebraic unification type of \mathcal{V} coincides with the unification type of \mathcal{V} .

Let us see now how these ideas relate to the notion of admissibility defined in the introduction. Recall that the *kernel* of a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ is defined as

$$\ker(h) = \{(a, b) \in A^2 \mid h(a) = h(b)\}.$$

In what follows, we will freely identify \mathcal{L} -identities with pairs of \mathcal{L} -formulas. We will also say that a \mathcal{L} -clause $\Sigma \Rightarrow \Delta$ is *valid* in a class of \mathcal{L} -algebras \mathcal{K} , written $\mathcal{K} \models \Sigma \Rightarrow \Delta$, if the universal sentence $(\forall \bar{x})(\bigwedge \Sigma \Rightarrow \bigvee \Delta)$ is valid in each algebra in \mathcal{K} .

Lemma 4. *Let $\Sigma \cup \Delta$ be a finite set of \mathcal{L} -identities. Then the following are equivalent:*

- (i) $\Sigma \Rightarrow \Delta$ is admissible in \mathcal{V} .

- (ii) $\mathbf{F}_\mathcal{V}(\omega) \models \Sigma \Rightarrow \Delta$.
- (iii) For each $\sigma: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$ such that $\Sigma \subseteq \ker(h_\mathcal{V} \circ \sigma)$,

$$\Delta \cap \ker(h_\mathcal{V} \circ \sigma) \neq \emptyset.$$

If in particular $\Delta = \{\varphi \approx \psi\}$, then (i)-(iii) above are also equivalent to

- (iv) $(\varphi, \psi) \in \bigcap \{\ker(h_\mathcal{V} \circ \sigma) \mid \sigma: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega) \text{ and } \Sigma \subseteq \ker(h_\mathcal{V} \circ \sigma)\}.$

Proof. (i) \Rightarrow (ii) Suppose that $\Sigma \Rightarrow \Delta$ is admissible in \mathcal{V} and let $g: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{F}_\mathcal{V}(\omega)$ be a homomorphism such that $\Sigma \subseteq \ker g$. Let σ be a map sending each variable x to a member of the equivalence class $g(x)$. By the universal mapping property for $\mathbf{Fm}_\mathcal{L}(\omega)$, this extends to a homomorphism $\sigma: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$. But $h_\mathcal{V}(\sigma(x)) = g(x)$ for each variable x , so $h_\mathcal{V} \circ \sigma = g$. Hence, for each $\varphi' \approx \psi' \in \Sigma$, also $h_\mathcal{V}(\sigma(\varphi')) = h_\mathcal{V}(\sigma(\psi'))$, i.e., $\mathcal{V} \models \sigma(\varphi') \approx \sigma(\psi')$. Therefore, σ is a unifier of Σ and, by assumption, $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$ for some $\varphi \approx \psi \in \Delta$. It follows that $g(\varphi) = h_\mathcal{V}(\sigma(\varphi)) = h_\mathcal{V}(\sigma(\psi)) = g(\psi)$ as required.

(ii) \Rightarrow (iii) Let $\sigma: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$ be such that $\Sigma \subseteq \ker(h_\mathcal{V} \circ \sigma)$, that is, $\mathcal{V} \models \sigma(\Sigma)$. Therefore $\mathbf{F}_\mathcal{V}(\omega) \models \sigma(\Sigma)$. By assumption, there exists an equation $\varphi \approx \psi \in \Delta$ such that $\mathbf{F}_\mathcal{V}(\omega) \models \sigma(\varphi) \approx \sigma(\psi)$, that is, $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$. Hence, $(\varphi, \psi) \in \ker(h_\mathcal{V} \circ \sigma) \cap \Delta$.

(iii) \Rightarrow (i) Let $\sigma: \mathbf{Fm}_\mathcal{L}(\omega) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$ be such that $\mathcal{V} \models \sigma(\Sigma)$, that is, $\Sigma \subseteq \ker(h_\mathcal{V} \circ \sigma)$. By hypothesis there exists $\varphi \approx \psi \in \Delta \cap \ker(h_\mathcal{V} \circ \sigma)$. Then $h_\mathcal{V}(\sigma(\varphi)) = h_\mathcal{V}(\sigma(\psi))$, i.e., $\mathcal{V} \models \sigma(\varphi) \approx \sigma(\psi)$. We obtained that $\Sigma \Rightarrow \Delta$ is admissible in \mathcal{V} .

If $\Delta = \{\varphi \approx \psi\}$, (iii) is equivalent to (iv). \square

Suppose now that \mathcal{V} is any equational class of \mathcal{L} -algebras and that S is a μ -set for the \preceq -preordered set of \mathcal{V} -unifiers of a finite set of \mathcal{L} -identities Γ . Then clearly:

$$\Gamma \Rightarrow \Delta \text{ is } \mathcal{V}\text{-admissible} \quad \Leftrightarrow \quad \text{each } \sigma \in S \text{ is a } \mathcal{V}\text{-unifier of some } (\varphi \approx \psi) \in \Delta.$$

Note in particular that if \mathcal{V} is unitary or finitary and there exists an algorithm for finding μ -sets, then checking admissibility in \mathcal{V} is decidable whenever the equational theory of \mathcal{V} is decidable. There are, however, many well-known equational classes having infinitary or nullary unification type, for which such a method is unavailable. The starting point for the new approach described below is the observation that the above equivalence can hold even when S is not a μ -set for the \preceq -preordered set of \mathcal{V} -unifiers. More precisely, it is enough that each $\sigma \in U_\mathcal{V}(\Gamma)$ \mathcal{V} -unifies all identities \mathcal{V} -unified by some particular member of S .

3. EXACT UNIFIERS

We begin by defining a new preorder on substitutions relative to a fixed equational class of \mathcal{L} -algebras \mathcal{V} . Let $X \subseteq \omega$ be a set of variables and let $\sigma_i: \mathbf{Fm}_\mathcal{L}(X) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$ be substitutions for $i = 1, 2$. We write $\sigma_2 \sqsubseteq_\mathcal{V} \sigma_1$ if all identities \mathcal{V} -unified by σ_1 are \mathcal{V} -unified by σ_2 . More precisely:

$$\sigma_2 \sqsubseteq_\mathcal{V} \sigma_1 \quad \Leftrightarrow \quad \ker(h_\mathcal{V} \circ \sigma_1) \subseteq \ker(h_\mathcal{V} \circ \sigma_2).$$

Clearly, \sqsubseteq is a preorder on substitutions of the form $\sigma: \mathbf{Fm}_\mathcal{L}(X) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$. Moreover:

Lemma 5. *Given $X \subseteq \omega$ and substitutions $\sigma_i: \mathbf{Fm}_\mathcal{L}(X) \rightarrow \mathbf{Fm}_\mathcal{L}(\omega)$ for $i = 1, 2$:*

$$\sigma_2 \preceq \sigma_1 \quad \Rightarrow \quad \sigma_2 \sqsubseteq_\mathcal{V} \sigma_1.$$

Proof. Suppose that $\sigma_2 \preceq \sigma_1$. Then there exists a substitution $\sigma': \mathbf{Fm}_{\mathcal{L}}(\omega) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ such that $\sigma' \circ \sigma_1 = \sigma_2$. But then if $h_{\mathcal{V}} \circ \sigma_1(\varphi) = h_{\mathcal{V}} \circ \sigma_1(\psi)$, also $h_{\mathcal{V}} \circ \sigma' \circ \sigma_1(\varphi) = h_{\mathcal{V}} \circ \sigma' \circ \sigma_1(\psi)$. That is, $h_{\mathcal{V}} \circ \sigma_2(\varphi) = h_{\mathcal{V}} \circ \sigma_2(\psi)$. \square

Given a finite set Σ of \mathcal{L} -identities and $X \supseteq \text{Var}(\Sigma)$, $\mathbf{E}_{\mathcal{V}}(\Sigma, X)$ is defined as the set of \mathcal{V} -unifiers of Σ over X preordered by $\sqsubseteq_{\mathcal{V}}$. For $X = \text{Var}(\Sigma)$, we simply write $\mathbf{E}_{\mathcal{V}}(\Sigma)$ instead of $\mathbf{E}_{\mathcal{V}}(\Sigma, X)$. Let us also define for $Y \subseteq X$ and a substitution $\sigma: \mathbf{Fm}_{\mathcal{L}}(Y) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$, the unique extension $\sigma_X: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ of σ as

$$\sigma_X(x) = \begin{cases} \sigma(x) & \text{if } x \in Y; \\ x & \text{otherwise.} \end{cases}$$

Lemma 6. *Let Σ be a finite set of identities and $X \supseteq \text{Var}(\Sigma)$. Then*

$$\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, X)) = \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma))$$

Proof. Let $Y = \text{Var}(\Sigma)$ and $(\cdot)|_Y: \mathbf{E}_{\mathcal{V}}(\Sigma, X) \rightarrow \mathbf{E}_{\mathcal{V}}(\Sigma, Y)$ be the map that assigns each unifier of Σ on X to its restriction to the variables in Y . It is easy to see that $(\cdot)|_Y$ preserves $\sqsubseteq_{\mathcal{V}}$. Let $(\cdot)_X: \mathbf{E}_{\mathcal{V}}(\Sigma, Y) \rightarrow \mathbf{E}_{\mathcal{V}}(\Sigma, X)$ be the map defined by $\sigma \rightarrow \sigma_X$. It is clear that $(\cdot)_X$ preserves $\sqsubseteq_{\mathcal{V}}$ and that $\sigma_X|_Y = \sigma$ for each $\sigma \in \mathbf{E}_{\mathcal{V}}(\Sigma, Y)$. This proves that $\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, Y)) = \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma)) \leq \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, X))$.

To see that $\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma)) \geq \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, X))$, let $\sigma \in \mathbf{E}_{\mathcal{V}}(\Sigma, X)$. Assume without loss of generality that $\text{Var}(\sigma(x)) \cap X = \emptyset$ for each $x \in X$. Define $\lambda: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ by

$$\lambda(x) = \begin{cases} x & \text{if } x \in Y; \\ \sigma(x) & \text{otherwise.} \end{cases}$$

Then $\sigma = \lambda \circ (\sigma|_Y)_X$, i.e., $\sigma \preceq (\sigma|_Y)_X$. Hence, if $S \subseteq \mathbf{E}_{\mathcal{V}}(\Sigma, Y)$ is a complete set, $\{\gamma_X \mid \gamma \in S\}$ is a complete set for $\mathbf{E}_{\mathcal{V}}(\Sigma, X)$. Thus $\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma)) \geq \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, X))$. \square

Suppose that Σ is a finite set of \mathcal{L} -identities and $\mathbf{E}_{\mathcal{V}}(\Sigma) \neq \emptyset$. Then the *exact type* of Σ in \mathcal{V} is defined as $\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma))$. We also define the *exact unification type* of Σ to be the maximal exact type of a \mathcal{V} -unifiable finite set of \mathcal{L} -identities.

Note that, because $\sigma_2 \preceq \sigma_1$ implies $\sigma_2 \sqsubseteq_{\mathcal{V}} \sigma_1$ (Lemma 5), every complete set for $\mathbf{U}_{\mathcal{V}}(\Sigma)$ is also a complete set for $\mathbf{E}_{\mathcal{V}}(\Sigma)$. Hence, for $\text{type}(\mathbf{U}_{\mathcal{V}}(\Sigma)) \in \{1, \omega\}$:

$$\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma)) \leq \text{type}(\mathbf{U}_{\mathcal{V}}(\Sigma)).$$

Moreover, using Lemmas 4 and 6 we obtain:

Corollary 7. *Let $\Sigma \cup \Delta$ be a finite set of \mathcal{L} -identities and S a complete set for $\mathbf{E}_{\mathcal{V}}(\Sigma)$. Then the following statements are equivalent:*

- (i) $\Sigma \Rightarrow \Delta$ is admissible in \mathcal{V} .
- (ii) For each $\sigma \in S$, the unifier $\sigma_X: \mathbf{Fm}_{\mathcal{L}}(\text{Var}(\Sigma \cup \Delta)) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ is a \mathcal{V} -unifier of some $\varphi \approx \psi \in \Delta$.
- (iii) For each $\sigma \in S$, $\Delta \cap \ker(h_{\mathcal{V}} \circ \sigma_X) \neq \emptyset$.

The close connection between exact types and admissible rules is also witnessed by the following result.

Theorem 8. *If an \mathcal{L} -clause $\Sigma \Rightarrow \Delta$ is \mathcal{V} -admissible and $\mathbf{E}_{\mathcal{V}}(\Sigma)$ has a finite μ -set S , then there exists $\Delta' \subseteq \Delta$ such that $|\Delta'| \leq |S|$ and $\Sigma \Rightarrow \Delta'$ is \mathcal{V} -admissible.*

Proof. Let $X = \text{Var}(\Sigma \cup \Delta)$ and let $\{\sigma_1, \dots, \sigma_n\}$ be a μ -set for $\mathbf{E}_V(\Sigma, X)$. By Lemma 6, $\text{type}(\mathbf{E}_V(\Sigma)) = \text{type}(\mathbf{E}_V(\Sigma, X))$. By Lemma 4, for each $i \in \{1, \dots, n\}$, there exists an identity $\varphi_i \approx \psi_i \in \Delta$ such that $(\varphi_i, \psi_i) \in \ker(h_V \circ \sigma_i)$. Let $\Delta' = \{\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n\}$. We claim that $\Sigma \Rightarrow \Delta'$ is admissible in V . Suppose that $\sigma: \mathbf{Fm}_L(X) \rightarrow \mathbf{Fm}_L(\omega)$ satisfies $\Sigma \subseteq \ker(h_V \circ \sigma)$. Since $\sigma \in \mathbf{E}_V(\Sigma, X)$ and $\{\sigma_1, \dots, \sigma_n\}$ is a μ -set for $\mathbf{E}_V(\Sigma, X)$, there exists $i \in \{1, \dots, n\}$ such that $\ker(h_V \circ \sigma_i) \subseteq \ker(h_V \circ \sigma)$. Hence $(\varphi_i, \psi_i) \in \ker(h_V \circ \sigma)$, and the result follows. \square

A finite set Σ of L -identities is said to be *admissibly reducible in V* if whenever $\Sigma \Rightarrow \Delta$ is admissible in V for some non-empty set of L -identities Δ , then there exists $\varphi \approx \psi \in \Delta$ such that $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in V .

Corollary 9. *Let Σ be a finite set of L -identities. If $\text{type}(\mathbf{E}_V(\Sigma)) = 1$, then Σ is admissibly reducible in V . Conversely, if $\text{type}(\mathbf{E}_V(\Sigma)) \in \{1, \omega\}$ and Σ is admissibly reducible in V then $\text{type}(\mathbf{E}_V(\Sigma)) = 1$.*

Proof. The first claim follows immediately from the previous theorem. For the second claim, assume that $\text{type}(\mathbf{E}_V(\Sigma)) \in \{\omega, 1\}$ and that Σ is admissibly reducible in V . Then there exists a μ -set $\{\sigma_1, \dots, \sigma_n\}$ for $\mathbf{E}_V(\Sigma)$. For each $i, j \in \{1, \dots, n\}$ such that $i \neq j$, consider $(\varphi_{ij}, \psi_{ij}) \in \ker(h_V \circ \sigma_i) \setminus \ker(h_V \circ \sigma_j)$. Let $\Delta = \{\varphi_{ij} \approx \psi_{ij} \mid i, j \in \{1, \dots, n\} \text{ and } i \neq j\}$.

Suppose that $n \neq 1$ and hence $\Delta \neq \emptyset$. Since $\{\sigma_1, \dots, \sigma_n\}$ is a μ -set for $\mathbf{E}_V(\Sigma)$, by Corollary 7, it follows that $\Sigma \Rightarrow \Delta$ is admissible in V . But, by assumption, there exists $\varphi_{ij} \approx \psi_{ij} \in \Delta$ such that $\Sigma \Rightarrow \varphi_{ij} \approx \psi_{ij}$ is admissible in V , contradicting the fact that $V \not\models \sigma_j(\varphi_{ij}) \approx \sigma_j(\psi_{ij})$. We conclude that $n = 1$, and hence that $\text{type}(\mathbf{E}_V(\Sigma)) = 1$. \square

4. ALGEBRAIC CO-EXACT UNIFIERS

We turn our attention now to the algebraic interpretation of exact unification. Following [12], a finite set of L -identities Σ will be called *exact in V* if there exists a substitution $\sigma: \mathbf{Fm}_L(\omega) \rightarrow \mathbf{Fm}_L(\omega)$ such that for all $\alpha, \beta \in \mathbf{Fm}_L(\text{Var}(\Sigma))$,

$$V \models \Sigma \Rightarrow \{\alpha \approx \beta\} \quad \Leftrightarrow \quad V \models \sigma(\alpha) \approx \sigma(\beta).$$

Note that by definition every exact set of identities is V -unifiable.

Given a finite set of L -identities Σ and a finite set of variables $X \supseteq \text{Var}(\Sigma)$, let $\rho_{(\Sigma, X, V)}: \mathbf{F}_V(X) \rightarrow \mathbf{Fp}_V(\Sigma, X)$ be the canonical quotient homomorphism from the free algebra $\mathbf{F}_V(X)$ to the finitely presented algebra $\mathbf{Fp}_V(\Sigma, X)$.

Lemma 10. *A finite set Σ of L -identities is exact in V if and only if*

$$\mathbf{Fp}_V(\Sigma, \text{Var}(\Sigma)) \in \mathbb{IS}(\mathbf{F}_V(\omega)).$$

Proof. (\Rightarrow) Let $X = \text{Var}(\Sigma)$ and let $\sigma: \mathbf{Fm}_L(\omega) \rightarrow \mathbf{Fm}_L(\omega)$ be a substitution such that for all $\alpha, \beta \in \mathbf{Fm}_L(X)$, $V \models \Sigma \Rightarrow \{\alpha \approx \beta\}$ iff $V \models \sigma(\alpha) \approx \sigma(\beta)$. That is $V \models \Sigma \Rightarrow \{\alpha \approx \beta\}$ iff $h_V(\sigma(\alpha)) = h_V(\sigma(\beta))$. There is a unique homomorphism $\sigma': \mathbf{F}_V(\omega) \rightarrow \mathbf{F}_V(\omega)$ such that $h_V \circ \sigma = \sigma' \circ h_V$ and hence $h_V(\Sigma) \subseteq \ker(\sigma')$.

Let $\iota: \mathbf{F}_V(X) \rightarrow \mathbf{F}_V(\omega)$ be the inclusion map. Since $h_V(\mathbf{Fm}_L(X)) = \mathbf{F}_V(X)$, it follows that $h_V(\Sigma) \subseteq \ker(\sigma' \circ \iota) = \ker(\sigma') \cap \mathbf{F}_V(X)^2$. There exists a unique $s: \mathbf{Fp}_V(\Sigma, X) \rightarrow \mathbf{F}_V(\omega)$ such that $s \circ \rho_{(\Sigma, X, V)} = \sigma' \circ \iota$. Let $a, b \in \mathbf{Fp}_V(\Sigma, X)$ be such that $s(a) = s(b)$ and $\alpha, \beta \in \mathbf{Fm}_L(X)$ such that $\rho_{(\Sigma, X, V)}(h_V(\alpha)) = a$ and $\rho_{(\Sigma, X, V)}(h_V(\beta)) = b$. Then

$$h_V \circ \sigma(\alpha) = \sigma' \circ h_V(\alpha) = (s \circ \rho_{(\Sigma, X, V)} \circ h_V)(\alpha) = s(a) = s(b)$$

$$(s \circ \rho_{(\Sigma, X, \mathcal{V})} \circ h_{\mathcal{V}})(\beta) = \sigma' \circ h_{\mathcal{V}}(\beta) = h_{\mathcal{V}} \circ \sigma_X(\beta).$$

By assumption, $\Sigma \models_{\mathcal{V}} \alpha \approx \beta$. So $a = \rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\alpha)) = \rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\beta)) = b$; i.e., s is a one-to-one homomorphism. Hence, $\mathbf{Fp}_{\mathcal{V}}(\Sigma, X) \in \mathbb{IS}(\mathbf{F}_{\mathcal{V}}(\omega))$.

(\Leftarrow) Let $X = \text{Var}(\Sigma)$, and let $s: \mathbf{Fp}_{\mathcal{V}}(\Sigma, X) \rightarrow \mathbf{F}_{\mathcal{V}}(\omega)$ be a one-to-one homomorphism. Let $\sigma: \mathbf{Fm}_{\mathcal{L}}(\omega) \rightarrow \mathbf{Fm}_{\mathcal{L}}(\omega)$ be the unique homomorphism determined by its value on the variables as follows:

$$\sigma(x) = \begin{cases} \alpha_x & \text{if } x \in X, \\ x & \text{otherwise,} \end{cases}$$

where α_x is any formula such that $s(\rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(x))) = h_{\mathcal{V}}(\alpha_x)$. By induction on formula complexity, $h_{\mathcal{V}}(\sigma(\varphi)) = s(\rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\varphi)))$ for each $\varphi \in \mathbf{Fm}_{\mathcal{L}}(X)$. Thus, if $\alpha, \beta \in \mathbf{Fm}_{\mathcal{L}}(X)$ are such that $h_{\mathcal{V}}(\sigma(\alpha)) = h_{\mathcal{V}}(\sigma(\beta))$, then $s(\rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\alpha))) = s(\rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\beta)))$. Finally from the injectivity of s it follows that $\rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\alpha)) = \rho_{(\Sigma, X, \mathcal{V})}(h_{\mathcal{V}}(\beta))$, equivalently, $\mathcal{V} \models \Sigma \Rightarrow \{\alpha \approx \beta\}$. \square

We call an algebra \mathbf{E} *exact* in \mathcal{V} if it is isomorphic to a finitely generated subalgebra of $\mathbf{F}_{\mathcal{V}}(\omega)$. By Lemma 10 (see also [12]), a finite set of identities Σ is exact iff the finitely presented algebra $\mathbf{Fp}_{\mathcal{V}}(\Sigma, \text{Var}(\Sigma))$ is exact.

Given $\mathbf{A} \in \mathbf{FP}(\mathcal{V})$, an onto homomorphism $u: \mathbf{A} \rightarrow \mathbf{E}$ is called a *coexact unifier* for \mathbf{A} if \mathbf{E} is exact. Coexact unifiers are ordered in the same way as algebraic unifiers, that is, if $u_i: \mathbf{A} \rightarrow \mathbf{E}_i$ for $i = 1, 2$ are coexact unifiers for \mathbf{A} , then $u_1 \leq u_2$, if there exists a homomorphism $f: \mathbf{E}_1 \rightarrow \mathbf{E}_2$ such that $f \circ u_1 = u_2$.

Let $\mathcal{C}_{\mathcal{V}}(\mathbf{A})$ be the set of coexact unifiers for \mathbf{A} preordered by \leq . If $\mathcal{C}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$, then the *exact type* of \mathbf{A} is defined as the type of $\mathcal{C}_{\mathcal{V}}(\mathbf{A})$. The *exact algebraic unification type* of \mathcal{V} is the maximal exact type of \mathbf{A} in \mathcal{V} such that $\mathcal{C}_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$.

We obtain the following Ghilardi-style result.

Theorem 11. *Let \mathcal{V} be an equational class and Σ a finite set of \mathcal{V} -unifiable \mathcal{L} -identities. Then for any $X \supseteq \text{Var}(\Sigma)$,*

$$\text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma)) = \text{type}(\mathbf{E}_{\mathcal{V}}(\Sigma, X)) = \text{type}(\mathcal{C}_{\mathcal{V}}(\mathbf{Fp}_{\mathcal{V}}(\Sigma, X))).$$

Hence the exact unification type and the exact algebraic unification type of \mathcal{V} coincide.

Proof. Consider $\sigma: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(Y)$ in $\mathbf{E}_{\mathcal{V}}(\Sigma, X)$. Let $\hat{\sigma}: \mathbf{F}_{\mathcal{V}}(X) \rightarrow h_{\mathcal{V}}(\sigma(\mathbf{Fm}_{\mathcal{L}}(X)))$ be the unique homomorphism determined by its value on the variables as follows:

$$\hat{\sigma}(h_{\mathcal{V}}(x)) = h_{\mathcal{V}}(\sigma(x)) \text{ for each } x \in X.$$

Then $\Sigma \subseteq \ker(\hat{\sigma} \circ h_{\mathcal{V}})$, and there exists a homomorphism $u_{\sigma}: \mathbf{Fp}_{\mathcal{V}}(\Sigma, X) \rightarrow h_{\mathcal{V}}(\mathbf{F}_{\mathcal{V}}(Y))$ such that

$$(1) \quad u_{\sigma} \circ \rho_{\Sigma, X, \mathcal{V}} = h_{\mathcal{V}} \circ \sigma.$$

Therefore, the map u_{σ} is onto $h_{\mathcal{V}}(\sigma(\mathbf{Fm}_{\mathcal{L}}(X)))$. Since $h_{\mathcal{V}}(\sigma(\mathbf{Fm}_{\mathcal{L}}(X)))$ is a finitely generated subalgebra of $\mathbf{F}_{\mathcal{V}}(Y)$, $u_{\sigma} \in \mathcal{C}_{\mathcal{V}}(\mathbf{Fp}_{\mathcal{V}}(\Sigma, X))$.

Let $u: \mathbf{Fp}_{\mathcal{V}}(\Sigma, X) \rightarrow \mathbf{E}$ be a coexact-unifier for $\mathbf{Fp}_{\mathcal{V}}(\Sigma, X)$. Since \mathbf{E} is exact, there exist some finite set Y and a one-to-one homomorphism $\iota: \mathbf{E} \rightarrow \mathbf{F}_{\mathcal{V}}(Y)$. For each $x \in X$, let $t_x \in \mathbf{Fm}_{\mathcal{L}}(Y)$ such that $h_{\mathcal{V}}(t_x) = \iota(u(\rho_{\Sigma, X, \mathcal{V}}(x)))$. Let $\sigma: \mathbf{Fm}_{\mathcal{L}}(X) \rightarrow \mathbf{Fm}_{\mathcal{L}}(Y)$ be the substitution defined by $\sigma(x) = t_x$ for each $x \in X$. It is straightforward to check that $\iota \circ u = u_{\sigma}$ and $\iota(\mathbf{E}) = u_{\sigma}(\mathbf{Fp}_{\mathcal{V}}(\Sigma, X))$. Since ι is one-to-one, there exists a homomorphism $\eta: u_{\sigma}(\mathbf{Fp}_{\mathcal{V}}(\Sigma, X)) \rightarrow \mathbf{E}$ that is the inverse of ι . Therefore u and u_{σ} are equivalent in the preorder $\mathcal{C}_{\mathcal{V}}(\mathbf{Fp}_{\mathcal{V}}(\Sigma, X))$.

By (1), for each $\sigma_1, \sigma_2 \in E_{\mathcal{V}}(\Sigma, X)$

$$\begin{aligned} \sigma_2 \sqsubseteq_{\mathcal{V}} \sigma_1 &\Leftrightarrow \ker(h_{\mathcal{V}} \circ \sigma_1) \subseteq \ker(h_{\mathcal{V}} \circ \sigma_2) \\ &\Leftrightarrow \ker(u_{\sigma_1} \circ \rho_{\Sigma, X, \mathcal{V}}) \subseteq \ker(\sigma_2 \circ \rho_{\Sigma, X, \mathcal{V}}) \\ &\Leftrightarrow \ker(u_{\sigma_1}) \subseteq \ker(u_{\sigma_2}). \end{aligned}$$

Let us denote the codomains of u_{σ_1} and u_{σ_2} by \mathbf{E}_1 and \mathbf{E}_2 , respectively. Since u_{σ_1} is onto \mathbf{E}_1 , $\ker(u_{\sigma_1}) \subseteq \ker(u_{\sigma_2})$ iff there exists $h: \mathbf{E}_1 \rightarrow \mathbf{E}_2$ such that $h \circ u_{\sigma_1} = u_{\sigma_2}$, that is $u_{\sigma_2} \leq u_{\sigma_1}$.

We have proved that the assignment $\sigma \mapsto u_{\sigma}$ determines an equivalence between the preorders $E_{\mathcal{V}}(\Sigma, X)$ and $C_{\mathcal{V}}(\mathbf{FP}_{\mathcal{V}}(\Sigma, X))$. Hence, the result follow by Lemma 1. \square

In the remainder of this section we present some consequences of the algebraic description of exact unification. Given an algebra \mathbf{A} in \mathcal{V} , let $\text{Con}_e(\mathbf{A})$ denote the set of congruences θ of \mathbf{A} such that the quotient \mathbf{A}/θ is exact; i.e.,

$$\text{Con}_e(\mathbf{A}) = \{\theta \in \text{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathbb{IS}(\mathbf{F}_{\mathcal{V}}(\omega))\}.$$

Theorem 12. *For any $\mathbf{A} \in \text{FP}(\mathcal{V})$:*

(i) *given any homomorphism $u: A \rightarrow B$,*

$$(u, u(\mathbf{A})) \in C_{\mathcal{V}}(\mathbf{A}) \Leftrightarrow \ker(u) \in \text{Con}_e(\mathbf{A}).$$

(ii) *$(u, \mathbf{B}), (v, \mathbf{C}) \in C_{\mathcal{V}}(\mathbf{A})$ are such that $u \leq v$ iff $\ker(v) \subseteq \ker(u)$.*

Hence $\ker: C_{\mathcal{V}}(\mathbf{A}) \rightarrow \text{Con}(\mathbf{A})$ determines an equivalence between the preordered set $C_{\mathcal{V}}(\mathbf{A})$ and the poset $(\text{Con}_e(\mathbf{A}), \supseteq)$.

Proof. (i) $(u, u(\mathbf{A})) \in C_{\mathcal{V}}(\mathbf{A})$ iff $u(\mathbf{A}) \in \mathbb{IS}(\mathbf{F}_{\mathcal{V}}(\omega))$ iff $\ker(u) \in \text{Con}_e(\mathbf{A})$.

(ii) $u \leq v$ iff there exists a homomorphism $f: \mathbf{C} \rightarrow \mathbf{B}$ such that $f \circ v = u$ iff (as v is surjective) $\ker(v) \subseteq \ker(u)$. \square

Corollary 13. *For each finitely presented algebra \mathbf{A} in \mathcal{V} ,*

$$\text{type}(C_{\mathcal{V}}(\mathbf{A})) = \begin{cases} 1 & \text{if } |\min(\text{Con}_e(\mathbf{A}))| = 1; \\ \omega & \text{if } 1 < |\min(\text{Con}_e(\mathbf{A}))| < \infty; \\ \infty & \text{if } \infty \leq |\min(\text{Con}_e(\mathbf{A}))|; \\ 0 & \text{if } \min(\text{Con}_e(\mathbf{A})) = \emptyset. \end{cases}$$

Corollary 14. *Let \mathcal{V} be a locally finite equational class. Then $\text{type}(C_{\mathcal{V}}(\mathbf{A}))$ is finite for each $\mathbf{A} \in \text{FP}(\mathcal{V})$. Hence \mathcal{V} has unitary or finitary exact unification type.*

Proof. As \mathcal{V} is locally finite, each finitely generated algebra in \mathcal{V} is finite. In particular \mathbf{A} is finite. Since $|\min(\text{Con}_e(\mathbf{A}))| \leq |\mathcal{P}(A \times A)| = 2^{|A|^2}$, where $\mathcal{P}(A \times A)$ denotes the powerset of $A \times A$, by Corollary 13, $\text{type}(C_{\mathcal{V}}(\mathbf{A}))$ is either unitary or finitary. \square

Corollary 15. *Let \mathbf{A} be a finitely presented algebra in \mathcal{V} such that its congruences are totally ordered. If $C_{\mathcal{V}}(\mathbf{A}) \neq \emptyset$, then it is totally ordered and $\text{type}(C_{\mathcal{V}}(\mathbf{A})) \in \{1, 0\}$. In particular, if \mathbf{A} is simple, then either $C_{\mathcal{V}}(\mathbf{A})$ is empty or $\text{type}(C_{\mathcal{V}}(\mathbf{A})) = 1$.*

Equational Class	Unification Type	Exact Type
Boolean Algebras	Unitary	Unitary
Heyting Algebras	Finitary	Finitary
Semigroups	Infinitary	Infinitary or Nullary
Modal algebras	Nullary	Nullary
Distributive Lattices	Nullary	Unitary
Stone Algebras	Nullary	Unitary
Bounded Distributive Lattices	Nullary	Finitary
Pseudocomplemented Distributive Lattices	Nullary	Finitary
Idempotent Semigroups	Nullary	Finitary
De Morgan Algebras	Nullary	Finitary
Kleene Algebras	Nullary	Finitary
MV-algebras	Nullary	Finitary
Willard's Example	Infinitary	Finitary

TABLE 1. Comparison of unification types and exact types

5. CASE STUDIES

Any unitary equational class such as the class of Boolean algebras also has exact unitary type, and any finitary equational class will have unitary or finitary exact type. For example, the class of Heyting algebras is finitary [14] and hence also has finitary exact type (the equation $x \vee y \approx \top$ has two most general exact unifiers as in Example 2). Minor changes to the original proofs that the class of semigroups has infinitary unification type [27] and that the class of modal algebras (for the logic K) has nullary unification type [18] establish that the former has infinitary or nullary exact type and the latter has nullary exact type. Below we consider more interesting cases where the type changes, collecting the results in Table 1.

Example 16 (Distributive Lattices). *However, the class of distributive lattices, which is known to have nullary unification type [13], has unitary exact type as all finitely presented distributive lattices are exact (see for example [10, Lemma 18]). The classes of bounded distributive lattices [13], idempotent semigroups (or bands) [1], De Morgan, and Kleene algebras [5] are also nullary, but because all these classes are locally finite, they have at most – and indeed, it can be shown via suitable cases, precisely – finitary exact type.*

Example 17 (Pseudocomplemented Distributive Lattices). *The equational class \mathfrak{B}_ω of pseudocomplemented distributive lattices is the class of algebras $(B, \wedge, \vee, *, 0, 1)$ such that $(B, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $a \wedge b^* = a$ if and only if $a \wedge b = 0$ for all $a, b \in B$. For each $n \in \mathbb{N}$, let $\mathbf{B}_n = (B_n, \wedge, \vee, *, 0, 1)$ denote the finite Boolean algebra with n atoms and let \mathbf{B}'_n be the algebra obtained by adding a new top $1'$ to the underlying lattice of \mathbf{B}_n and endowing it with the unique operation making it into a pseudocomplemented distributive lattice. Let \mathfrak{B}_n denote the subvariety of \mathfrak{B}_ω generated by \mathbf{B}'_n . Lee proved in [24], that the subvariety lattice of \mathfrak{B}_ω is*

$$\mathfrak{B}_0 \subsetneq \mathfrak{B}_1 \subsetneq \cdots \subsetneq \mathfrak{B}_n \subsetneq \cdots \subsetneq \mathfrak{B}_\omega$$

where \mathfrak{B}_0 and \mathfrak{B}_1 are the varieties of Boolean algebras and Stone algebras, respectively. We have already observed that the class of Boolean algebras has exact type 1. The case of Stone algebras is similar to distributive lattices: \mathfrak{B}_1 has nullary unification [13] type; however, all finitely presented Stone algebras are exact (see [10, Lemma 20]), so the class of Stone algebras has unitary exact type.

In [13] it was proved that \mathfrak{B}_ω has nullary unification type, and the same result was proved in [8] for \mathfrak{B}_n for each $n \geq 2$. All these varieties are locally finite, so an application of Corollary 14 proves that they have at most finitary unification type. It is easy to prove that $x \vee \neg x \approx \top \Rightarrow x \approx \top, \neg x \approx \perp$ is admissible in \mathfrak{B}_ω and \mathfrak{B}_n for each $n \geq 2$ and that neither $x \vee \neg x \approx \top \Rightarrow x \approx \top$ nor $x \vee \neg x \approx \top \Rightarrow \neg x \approx \perp$ are admissible in \mathfrak{B}_ω or \mathfrak{B}_n with $n \geq 2$. By Corollary 15, the classes \mathfrak{B}_ω and \mathfrak{B}_n with $n \geq 2$ have finitary type.

Example 18 (A Locally Finite Equational Class with Infinitary Unification Type). The following example of a locally finite equational class with infinitary unification type is due to R. Willard (private communication). Consider a language with one binary operation, written as juxtaposition, and two constants 0 and 1. Let \mathcal{V} be the equational class defined by

$$0x \approx x0 \approx 0, \quad 1x \approx 0, \quad x(yz) \approx 0, \quad (x1)1 \approx x1,$$

and, for each $n \in \mathbb{N}$, associating to the left,

$$xyz_1z_2 \dots z_n y \approx xyz_1z_2 \dots z_n 1.$$

Then up to equivalence, terms have the form (again associating to the left)

$$0, \quad 1, \quad \text{or} \quad xy_1y_2 \dots y_n$$

where y_1, \dots, y_n are variables or 1 and all distinct, and x is any variable. It is immediate that finitely generated free algebras are finite and hence that \mathcal{V} is locally finite. Note also that $\{xy \approx 0\}$ has three most general exact unifiers

$$\sigma_1(x) = 1, \sigma_1(y) = y; \quad \sigma_2(x) = 0, \sigma_2(y) = y; \quad \sigma_3(x) = x, \sigma_3(y) = yz.$$

So the exact unification type of \mathcal{V} is finitary.

We now claim that the following set of identities has infinitary unification type:

$$\Sigma = \{xy \approx x1\}.$$

For each $n \in \mathbb{N}$ and distinct variables z_1, \dots, z_n different from y , consider the following \mathcal{V} -unifier of Σ :

$$\sigma_n(x) = xyz_1 \dots z_n, \quad \sigma_n(y) = y.$$

Then the set $\{\sigma_n \mid n \in \mathbb{N}\}$ is a μ -set for $\cup_{\mathcal{V}}(\Sigma)$. Moreover, it can be shown that no set of identities has nullary unification type.

Example 19 (MV-algebras). In [25] it is proved that the equational class \mathcal{MV} of MV-algebras has nullary unification type. This class is not locally finite, so we cannot apply Corollary 14. However, combining results from [21] and [9], we can still prove that MV-algebras have finitary exact type.

Let \mathcal{L} be the language of MV-algebras and Σ a finite set of equations in $\mathbf{Fm}_{\mathcal{L}}(\omega)$. Finitely presented MV-algebras admit a presentation of the form $\{\alpha \approx \top\}$, so there is no loss of generality in assuming that $\Sigma = \{\alpha \approx \top\}$. Let us fix $X = \text{Var}(\alpha)$ and $\mathbf{A} = \mathbf{Fp}_{\mathcal{MV}}(\{\alpha \approx \top\})$. A combination of [21, Theorem 3.8] and [9,

Theorem 4.18] proves the following result. There exist $\beta_1, \dots, \beta_n \in \mathbf{Fm}_{\mathcal{L}}(X)$ such that the following hold:

- (i) the rule $\{\alpha \approx \top\} \Rightarrow \{\beta_1 \approx \top, \dots, \beta_n \approx \top\}$ is admissible in \mathcal{MV} ;
- (ii) $\{\beta_i \approx \top\} \models_{\mathcal{MV}} \alpha \approx \top$ for each $i \in \{1, \dots, n\}$;
- (iii) $\mathbf{Fp}_{\mathcal{MV}}(\{\beta_i \approx \top\})$ is exact for each $i \in \{1, \dots, n\}$.

Defining $\mathbf{B}_i = \mathbf{Fp}_{\mathcal{MV}}(\{\beta_i \approx \top\})$, from (ii), we obtain that for each $i \in \{1, \dots, n\}$, there exists a homomorphism $e_i: \mathbf{A} \rightarrow \mathbf{B}_i$ such that $\rho_{\{\beta_i \approx \top\}, X, \mathcal{MV}} = e_i \circ \rho_{\{\alpha \approx \top\}, X, \mathcal{MV}}$. Since $\rho_{\{\beta_i \approx \top\}, X, \mathcal{MV}}$ is onto, so is e_i . By (iii), it follows that $S = \{e_1, \dots, e_n\}$ is a set of coexact unifiers of \mathbf{A} . We claim that S is a complete set in $\mathbf{C}_{\mathcal{MV}}(\mathbf{A})$. Indeed, let $e: \mathbf{A} \rightarrow \mathbf{C} \in \mathbf{C}_{\mathcal{MV}}(\mathbf{A})$. By (i), there exists $i \in \{1, \dots, n\}$ and $h: \mathbf{B}_i \rightarrow \mathbf{C}$ such that $e \circ \rho_{\{\alpha \approx \top\}, X, \mathcal{MV}} = h \circ \rho_{\{\beta_i \approx \top\}, X, \mathcal{MV}}$. Since $\rho_{\{\alpha \approx \top\}, X, \mathcal{MV}}$ is onto and $\rho_{\{\beta_i \approx \top\}, X, \mathcal{MV}} = e_i \circ \rho_{\{\alpha \approx \top\}, X, \mathcal{MV}}$, it follows that $e = h \circ e_i$, that is, $e \leq e_i$. This proves that $\text{type}(\mathbf{C}_{\mathcal{MV}}(\mathbf{A})) \in \{1, \omega\}$, hence the exact type of \mathcal{MV} is either unitary or finitary. By [21, Lemma 4.2], $x \vee \neg x \approx \top \Rightarrow x \approx \top, \neg x \approx \perp$ is admissible in \mathcal{MV} and it is easy to see that neither $x \vee \neg x \approx \top \Rightarrow x \approx \top$ nor $x \vee \neg x \approx \top \Rightarrow \neg x \approx \perp$ are admissible. So by Corollary 15, \mathcal{MV} has finitary exact type. It is possible to prove that $\text{type}(\mathbf{C}_{\mathcal{MV}}(\mathbf{Fp}_{\mathcal{MV}}(\{x \vee \neg x \approx \top\}))) = 2$, but such a calculation is beyond the scope of this paper.

6. CONCLUDING REMARKS

We have introduced a new hierarchy of exact unification types based on an inclusion preordering of unifiers, showing that in certain cases, the exact type reduces from nullary or infinitary unification type to finitary or even unitary exact type. Note, however, that we do not know if there are examples of equational classes of (i) finitary unification type that have unitary exact type, (ii) infinitary unification type that have unitary or nullary exact type, (iii) nullary unification type that have infinitary exact type.

In [10], the current authors present axiomatizations for admissible rules of several locally finite (and hence of finitary exact unification type) equational classes with classical unification type 0. In all these cases a complete description of exact algebras, and the finite exact unification type plays a central (if implicit) role. We therefore expect that this approach will be useful for tackling other classes of algebras that have unitary or finitary exact type, independently of their unification type.

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